Tentamen Analyse 2013

Date	:	04-11-2013
Time	:	14.00 - 17.00
Place	:	Aletta Jacobshal 02

Provide clear arguments for all your answers; 'yes' or 'no' answers are not allowed. In your argumentation you may use all theorems and statements in the book. However, you should indicate which theorems/ statements you are using.

The detailed grading scheme can be found below.

1. Let $f : (0, 1) \to \mathbb{R}$ be differentiable, with $|f'(x)| \leq M$ for all $x \in (0, 1)$ for some positive constant M.
   (a) Let the sequence (x_n), $x_n > 0$, be such that $(x_n) \to 0$. Prove that $\lim_{n\to\infty} f(x_n)$ exists.
   Hint: Show that $(f(x_n))$ is a Cauchy sequence.
   (b) Conclude that $\lim_{x\to 0^+} f(x)$ exists.

2. (a) Let $f : [0, \infty) \to \mathbb{R}$ be continuous. Prove (by using theorem(s) from the book) that for every $c > 0$ the restricted function $f : [0, c] \to \mathbb{R}$ is uniformly continuous.
   (b) Formulate the Sequential Criterion for Nonuniform Continuity of a function $f : [0, \infty) \to \mathbb{R}$.
   (c) Prove by use of the Sequential Criterion for Nonuniform Continuity that the continuous function $f : [0, \infty) \to \mathbb{R}$ defined as $f(x) = x^2$ is not uniformly continuous.
   Hint: Consider the sequences $(x_n), (y_n)$ defined as $x_n = n, y_n = n + \frac{1}{n}$.
   (d) Let $f : [0, \infty) \to \mathbb{R}$ be continuous *and* such that $\lim_{x\to\infty} f(x)$ exists. Prove that f is uniformly continuous on $[0, \infty)$.

3. Consider a continuous function $f : [0, 1] \to \mathbb{R}$ with $0 \leq f(x) \leq 1$ for all $x \in [0, 1]$.
   (a) Prove that there exists at least one $x \in [0, 1]$ with $f(x) = x^2$.
   Hint: Consider the function $f(x) - x^2$.
   (b) Suppose additionally that f is differentiable on $(0, 1)$ and that $f(0) = 0, f(1) = 1$. Show that there exists an $x \in (0, 1)$ such that $f'(x) = 2x$.

4. Consider the infinite series

$$\sum_{n=0}^{\infty}(-1)^n x^n(1-x), \quad x \in [0,1]$$

   (a) Show by computation that this series converges pointwise on $[0,1]$ to the function $s(x) = \frac{1-x}{1+x}$.
   (**Hint**: Recall that the n-th partial sum s_n of the geometric series $\sum_{n=0}^{\infty} r^n$ is given by $s_n = \frac{1-r^n}{1-r}$.)

   (b) Show that the convergence of the series to the function s is uniform on any interval $[0,\delta]$ with $0 < \delta < 1$.

   (c) Show that the convergence of the series to the function s is uniform on the whole interval $[0,1]$.

   (d) Compute the pointwise limit of the series of *absolute values* for any $x \in [0,1]$. Is this convergence also uniform on $[0,1]$? Explain your answer.

5. Consider two Riemann integrable functions $f, g : [a,b] \to \mathbb{R}$, satisfying $f(x) \leq g(x)$ for all $x \in [a,b]$. It is well-known that this implies that $\int_a^b f \leq \int_a^b g$.
   Give a direct proof of this fact which is solely based on the definition of the Riemann integral.

Grading scheme:

Total 100, Free 10.

1. a: 8, b: 4.
2. a: 5, b: 4, c: 8, d: 8.
3. a: 8, b: 8.
4. a: 5, b: 8, c: 7, d: 7.
5. 10.